MTH-129 Answers to Review for Test 2 Luczak

Please note that some steps have been left out (mostly algebraic) that I would expect to see on a test.

1. Since $a \mid b$ means $b = ar,\ r \in \mathbb{Z}$ and $a \mid c$ means $c = as,\ s \in \mathbb{Z}$ then

   $$\begin{aligned} 5b + 3c &= 5(ar) + 3(as) \text{ by substitution} \\ &= a(5r + 3s) \text{ by algebra} \end{aligned}$$

   and $5r + 3s$ must be an integer by closure of integers. Therefore, by definition of divisibility $a \mid (5b + 3c)$.
2. Case 1: Consider the case where n is even.

   Let $n = 2r,\ r \in \mathbb{Z}$ then

   $$\begin{aligned} n^2 + n + 1 &= (2r)^2 + (2r) + 1 \\ &= 2(2r^2 + r) + 1 \end{aligned}$$

Since $(2r^2 + r) \in \mathbb{Z}$ by closure $n^2 + n + 1$ is odd by definition.

Case 2: Consider the case where n is odd.
Let $n = 2r + 1,\ r \in \mathbb{Z}$ then

$$\begin{aligned} n^2 + n + 1 &= (2r + 1)^2 + (2r + 1) + 1 \\ &= 2(2r^2 + 3r + 1) + 1 \end{aligned}$$

Since $(2r^2 + 3r + 1) \in \mathbb{Z}$ by closure $n^2 + n + 1$ is odd by definition.

3. Since $a \mid b$ by definition $b = ar,\ r \in \mathbb{Z}$ thus $\begin{aligned} b^2 &= (ar)^2 \\ &= a^2 r^2 \end{aligned}$ where $r^2 \in \mathbb{Z}$ by closure. Thus $a^2 \mid b^2$ by definition.
4. Let $a = 6, b = 4, c = 3$.
5. $n = 2^2 \cdot 3^2 \cdot 7 = 252$
6. $3^2 \cdot 7^2 \cdot 13$
7. Assumes what is to be proved is true without actually proving it is true. (Jumping to a conclusion.)
8. a) $q = 3, r = 4$ b) $q = -4, r = 7$
9. 0
10. Let $m = 5q + 2$ and $n = 5r + 1$ by definition. Then $\begin{aligned} mn &= (5q + 2)(5r + 1) \\ &= 5(5qr + q + 2r) + 2 \end{aligned}$ Where $5qr + q + 10r \in \mathbb{Z}$ by closure of integers. Therefore by definition $mn \bmod 5 = 2$.
11. Case 1: $n = 3r$ $(3r)(3r + 1) = 3(3r^2 + r)$, let $k = 3r^2 + r$

Case 2: $n=3r+1$ $(3r+1)(3r+2)=3\left(3r^2+3r\right)+2$, let $k=3r^2+3r$

Case 3: $n=3r+2$ $(3r+2)(3r+3)=3\left(3r^2+5r+2\right)$, let $k=3r^2+5r+2$.

12. The number must be an integer.
13. a) -7 b) -4
14. $\left\lceil \frac{c}{m} \right\rceil$
15. **Proof by contradiction**: Suppose there is an integer n such that n^3 is even and n is odd. Let $n=2k+1$ then $n^3=(2k+1)^3=2\left(4k^3+6k^2+3k\right)+1$ let $t=4k^3+6k^2+3k, t\in\mathbb{Z}$ by closure. Thus $n^3=2t+1$ is odd by definition, which is a contradiction. Therefore, statement is true.
Proof by contraposition: Suppose n is an odd integer, so let $n=2k+1$ then $n^3=(2k+1)^3=2\left(4k^3+6k^2+3k\right)+1$ let $t=4k^3+6k^2+3k, t\in\mathbb{Z}$ by closure. Thus $n^3=2t+1$ is odd by definition. Therefore, proving the contrapositive is true thus the statement itself is true.
16. Suppose there are integers m and n $\ni$ mn is even and m and n are both odd. Let $m=2k+1$ and $n=2r+1$ then $mn=2\left(2kr+k+r\right)+1$ letting $t=2kr+k+r, t\in\mathbb{Z}$ by closure we have $mn=2t+1$ which is an odd integer by definition and therefore a contradiction.
17. Suppose $\exists r,s\in\mathbb{R}\ni r\in\mathbb{Q}$ and $s\notin\mathbb{Q}$ and $r+2s$ is rational. By definition $r=\dfrac{a}{b}$ and $r+2s=\dfrac{c}{d}$ with $a,b,c,d\in\mathbb{Z}$ and $b\neq 0$ and $d\neq 0$. Then using substitution and algebra we arrive at $s=\dfrac{bc-ad}{2bd}$ with $bc-ad\in\mathbb{Z}$ and $2bd\in\mathbb{Z}$ and $2bd\neq 0$ making s rational by definition which is a contradiction.
18. False, all integers are rational numbers so consider $x=4$.
19. a) $\sum_{k=2}^{n}(-3)^k$ b) $\prod_{n=2}^{6}\left(n-t^{n-1}\right)$
20. a) $\dfrac{n(n+1)(n+2)}{6}$ b) $(n+1)^2$
21. a) $\sum_{k=m}^{n}\left(a_k-cb_k\right)$ b) $\prod_{k=m}^{n}a_k b_k$
22. $P(1)$ $1^3=\dfrac{1^2(1+1)^2}{4}$

$1=1$

$P(k)$ $\displaystyle\sum_{i=1}^{k}i^3=\frac{k^2(k+1)^2}{4}$

Need to show $P(k+1)$ $\displaystyle\sum_{i=1}^{k+1}i^3=\frac{(k+1)^2(k+2)^2}{4}$

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

LHS

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$$

RHS

$$\frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$$

Thus $P(k+1)$ has been shown to be true.

23. $P(1)$

$$\frac{1}{1\cdot 2} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

$P(k)$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Need to show $P(k+1)$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$$

LHS

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)((k+1)+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

RHS $\dfrac{k+1}{(k+1)+1} = \dfrac{k+1}{k+2}$

Thus $P(k+1)$ has been shown to be true.

24. $P(0)$ $$1=\frac{3^{0+1}-1}{2}$$
$$=1$$

$P(k)$ $$1+3+3^2+\ldots+3^k=\frac{3^{k+1}-1}{2}$$

Need to show $P(k+1)$ $$1+3+3^2+\ldots+3^k+3^{k+1}=\frac{3^{(k+1)+1}-1}{2}$$

LHS $$1+3+3^2+\ldots+3^k+3^{k+1}=\frac{3^{k+1}-1}{2}+3^{k+1}$$
$$=\frac{3^{k+2}-1}{2}$$

RHS $$\frac{3^{(k+1)+1}-1}{2}=\frac{3^{k+2}-1}{2}$$

Thus $P(k+1)$ has been shown to be true.

25. $P(0)$ $7\mid 0$

$P(k)$ $7\mid(8^k-1)$ by definition of divisibility $8^k-1=7r,\ r\in\mathbb{Z}$

Need to show $P(k+1)$ $7\mid(8^{k+1}-1)$ this would mean that $8^{k+1}-1=7s,\ s\in\mathbb{Z}$.

LHS
$$\begin{aligned}8^{k+1}-1&=8\cdot 8^k-8+7\\&=8(8^k-1)+7\\&=8(7r)+7 \qquad \text{let } 8r+1=s\\&=7(8r+1)\\&=7s\end{aligned}$$

26. $P(3)$ $2(3)+1<2^3$

$P(k)$ $2k+1<2^k$

Need to show $P(k+1)$ $$2(k+1)+1<2^{k+1}$$
$$2k+1+2<2^k+2<2\cdot 2^k=2^{k+1}$$